Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2015

Mathematics

Higher level

Paper 1

22 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).*
*A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \big(2\cos(5x-3)\big)5 \quad \big(= 10\cos(5x-3)\big) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\big(2\cos(5x-3)\big)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) **METHOD 1**

area $=\pi 2^2-\frac{1}{2}2^2\theta\left(=3\pi\right)$ ***M1A1***

Note: Award ***M1*** for using area formula.

$\Rightarrow 2\theta=\pi\Rightarrow\theta=\frac{\pi}{2}$ ***A1***

Note: Degrees loses final A1

METHOD 2

let $x=2\pi-\theta$

area $=\frac{1}{2}2^2x\,(=3\pi)$ ***M1***

$\Rightarrow x=\frac{3}{2}\pi$ ***A1***

$\Rightarrow\theta=\frac{\pi}{2}$ ***A1***

METHOD 3

Area of circle is 4π ***A1***

Shaded area is $\frac{3}{4}$ of the circle ***(R1)***

$\Rightarrow\theta=\frac{\pi}{2}$ ***A1***

[3 marks]

(b) arc length $=2\frac{3\pi}{2}$ ***A1***

perimeter $=2\frac{3\pi}{2}+2\times 2$

$=3\pi+4$ ***A1***

[2 marks]

Total [5 marks]

2. (a) $\bar{x} = \dfrac{1\times 0 + 19\times 10}{20} = 9.5$ ***(M1)A1***

[2 marks]

(b) median is 10 ***A1***

[1 mark]

(c) (i) 19 ***A1***

(ii) 1 ***A1***

[2 marks]

Total [5 marks]

3. (a) $\int\left(1+\tan^2 x\right)\mathrm{d}x = \int \sec^2 x\,\mathrm{d}x = \tan x\,(+c)$ ***M1A1***

[2 marks]

(b) $\int \sin^2 x\,\mathrm{d}x = \int \dfrac{1-\cos 2x}{2}\mathrm{d}x$ ***M1A1***

$= \dfrac{x}{2} - \dfrac{\sin 2x}{4}(+c)$ ***A1***

Note:	Allow integration by parts followed by trig identity. Award ***M1*** for parts, ***A1*** for trig identity, ***A1*** final answer.

[3 marks]

Total [5 marks]

4. (a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ ***(M1)A1***

[2 marks]

(b) $f'(x) = \lim\limits_{h\to 0}\dfrac{(x+h)^3 - x^3}{h}$ ***(M1)***

$= \lim\limits_{h\to 0}\dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim\limits_{h\to 0}\left(3x^2 + 3xh + h^2\right)$ ***A1***

$= 3x^2$ ***A1***

Note: Do not award final A1 on FT if $= 3x^2$ is not obtained

Note: Final A1 can only be obtained if previous A1 is given

[3 marks]

Total [5 marks]

5. (a) **EITHER**

$f(-x) = f(x)$ ***M1***

$\Rightarrow ax^2 - bx + c = ax^2 + bx + c \Rightarrow 2bx = 0, \ (\forall x \in \mathbb{R})$ ***A1***

OR

y-axis is eqn of symmetry ***M1***

so $\frac{-b}{2a} = 0$ ***A1***

THEN

$\Rightarrow b = 0$ ***AG***

[2 marks]

(b) $g(-x) = -g(x) \Rightarrow p\sin(-x) - qx + r = -p\sin x - qx - r$

$\Rightarrow -p\sin x - qx + r = -p\sin x - qx - r$ ***M1***

Note: ***M1*** is for knowing properties of sin.

$\Rightarrow 2r = 0 \Rightarrow r = 0$ ***A1***

Note: In (a) and (b) allow substitution of a particular value of x

[2 marks]

(c) $h(-x) = h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0, \ (\forall x)$ ***M1A1***

Note: Accept geometrical explanations.

[2 marks]

Total [6 marks]

6. (a) $f: x \to y = \dfrac{3x-2}{2x-1}$ $f^{-1}: y \to x$

$y = \dfrac{3x-2}{2x-1} \Rightarrow 3x - 2 = 2xy - y$ ***M1***

$\Rightarrow 3x - 2xy = -y + 2$ ***M1***

$x(3-2y) = 2 - y$

$x = \dfrac{2-y}{3-2y}$ ***A1***

$\left(f^{-1}(y) = \dfrac{2-y}{3-2y}\right)$

$f^{-1}(x) = \dfrac{2-x}{3-2x}$ $\left(x \neq \dfrac{3}{2}\right)$ ***A1***

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

[4 marks]

(b) $\dfrac{3x-2}{2x-1} = A + \dfrac{B}{2x-1} \Rightarrow 3x - 2 = A(2x-1) + B$

equating coefficients $3 = 2A$ and $-2 = -A + B$ ***(M1)***

$A = \dfrac{3}{2}$ and $B = -\dfrac{1}{2}$ ***A1***

Note: Could also be done by division or substitution of values.

[2 marks]

(c) $\int f(x)\,\mathrm{d}x = \dfrac{3}{2}x - \dfrac{1}{4}\ln|2x-1| + c$ ***A1***

Note: accept equivalent e.g. ln $|4x-2|$

[1 mark]

Total [7 marks]

7. (a) (i) $\left(-\frac{a_{n-1}}{a_n}=\right)-\frac{1}{2}$ ***A1***

(ii) $\left((-1)^n\frac{a_0}{a_n}=\right)-\frac{36}{2}=(-18)$ ***A1A1***

Note: First ***A1*** is for the negative sign.

[3 marks]

(b) **METHOD 1**

if λ satisfies $p(\lambda)=0$ then $q(\lambda-4)=0$

so the roots of $q(x)$ are each 4 less than the roots of $p(x)$ ***(R1)***

so sum of roots is $-\frac{1}{2}-4\times5=-20.5$ ***A1***

METHOD 2

$p(x+4)=2x^5+2\times5\times4x^4\ldots+x^4\ldots=2x^5+41x^4\ldots$ ***(M1)***

so sum of roots is $-\frac{41}{2}=-20.5$ ***A1***

[2 marks]

Total [5 marks]

8. $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ ***(A1)***

EITHER

integral is $\int \frac{\mathrm{e}^x}{\left(\mathrm{e}^x + 3\right)^2 + 2^2}\,\mathrm{d}x$ ***M1A1***

$= \int \frac{1}{u^2 + 2^2}\,\mathrm{d}u$ ***M1A1***

Note:	Award ***M1*** only if the integral has completely changed to one in u.

Note:	$\mathrm{d}u$ needed for final ***A1***

OR

$\mathrm{e}^x = u - 3$

integral is $\int \frac{1}{(u-3)^2 + 6(u-3) + 13}\,\mathrm{d}u$ ***M1A1***

Note:	Award ***M1*** only if the integral has completely changed to one in u.

$= \int \frac{1}{u^2 + 2^2}\,\mathrm{d}u$ ***M1A1***

Note:	In both solutions the two method marks are independent.

THEN

$= \frac{1}{2}\arctan\left(\frac{u}{2}\right)(+c)$ ***(A1)***

$= \frac{1}{2}\arctan\left(\frac{\mathrm{e}^x + 3}{2}\right)(+c)$ ***A1***

Total [7 marks]

9. (a) $g \circ f(x) = g(f(x))$ ***M1***

$= g\left(2x + \frac{\pi}{5}\right)$

$= 3\sin\left(2x + \frac{\pi}{5}\right) + 4$ ***AG***

[1 mark]

(b) since $-1 \le \sin\theta \le +1$, range is $[1, 7]$ ***(R1)A1***

[2 marks]

(c) $3\sin\left(2x + \frac{\pi}{5}\right) + 4 = 7 \Rightarrow 2x + \frac{\pi}{5} = \frac{\pi}{2} + 2n\pi \Rightarrow x = \frac{3\pi}{20} + n\pi$ ***(M1)***

so next biggest value is $\frac{23\pi}{20}$ ***A1***

Note: Allow use of period.

[2 marks]

(d) **Note:** Transformations can be in any order but see notes below.

stretch scale factor 3 parallel to y axis (vertically) ***A1***
vertical translation of 4 up ***A1***

Note: Vertical translation is $\frac{4}{3}$ up if it occurs before stretch parallel to y axis.

stretch scale factor $\frac{1}{2}$ parallel to x axis (horizontally) ***A1***

horizontal translation of $\frac{\pi}{10}$ to the left ***A1***

Note: Horizontal translation is $\frac{\pi}{5}$ to the left if it occurs before stretch parallel to x axis.

Note: Award ***A1*** for magnitude and direction in each case. Accept any correct terminology provided that the meaning is clear *eg* shift for translation.

[4 marks]

Total [9 marks]

10. METHOD 1

to have 3 consecutive losses there must be exactly 5, 4 or 3 losses

the probability of exactly 5 losses (must be 3 consecutive) is $\left(\frac{1}{3}\right)^5$ ***A1***

the probability of exactly 4 losses (with 3 consecutive) is $4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)$ ***A1A1***

Note: First ***A1*** is for the factor 4 and second ***A1*** for the other 2 factors.

the probability of exactly 3 losses (with 3 consecutive) is $3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$ ***A1A1***

Note: First ***A1*** is for the factor 3 and second ***A1*** for the other 2 factors.

(Since the events are mutually exclusive)
the total probability is $\frac{1+8+12}{3^5}=\frac{21}{243}\left(=\frac{7}{81}\right)$ ***A1***

[6 marks]

METHOD 2

Roy loses his job if
A – first 3 games are all lost (so the last 2 games can be any result)
B – first 3 games are not all lost, but middle 3 games are all lost (so the first game is not a loss and the last game can be any result)
or C – first 3 games are not all lost, middle 3 games are not all lost but last 3 games are all lost, (so the first game can be any result but the second game is not a loss)

for A 4^{th} & 5^{th} games can be anything

$$\mathrm{P}(A)=\left(\frac{1}{3}\right)^3=\frac{1}{27}$$ ***A1***

for B 1^{st} game not a loss & 5^{th} game can be anything ***(R1)***

$$\mathrm{P}(B)=\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$$ ***A1***

for C 1^{st} game anything, 2^{nd} game not a loss ***(R1)***

$$\mathrm{P}(C)=1\times\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$$ ***A1***

(Since the events are mutually exclusive)

total probability is $\frac{1}{27}+\frac{2}{81}+\frac{2}{81}=\frac{7}{81}$ ***A1***

continued…

Question 10 continued.

Note:	In both methods all the ***A*** marks are independent.

Note:	If the candidate misunderstands the question and thinks that it is asking for exactly 3 losses award ***A1 A1*** and ***A1*** for an answer of $\frac{12}{243}$ as in the last lines of Method 1.

[6 marks]

Total [6 marks]

Section B

11. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = \left(\mathrm{e}^{3x} + 3x\mathrm{e}^{3x}\right)$ ***M1A1***

[2 marks]

(b) let $P(n)$ be the statement $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ which is $1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x}$ and RHS is $3^0\mathrm{e}^{3x} + x3^1\mathrm{e}^{3x}$ ***R1***

as LHS = RHS, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = k3^{k-1}\mathrm{e}^{3x} + x3^k\mathrm{e}^{3x}$

$\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)$ ***(M1)***

$= k3^{k-1} \times 3\mathrm{e}^{3x} + 1 \times 3^k\mathrm{e}^{3x} + x3^k \times 3\mathrm{e}^{3x}$ ***A1***

$= (k+1)3^k\mathrm{e}^{3x} + x3^{k+1}\mathrm{e}^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true ($\forall n \in \mathbb{Z}^+$) ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

continued…

Question 11 continued

(c) $e^{3x} + x \times 3e^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ ***M1A1***

point is $\left(-\frac{1}{3}, -\frac{1}{3e}\right)$ ***A1***

EITHER

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$

when $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} > 0$ therefore the point is a minimum ***M1A1***

OR

x	$-\frac{1}{3}$
$\frac{dy}{dx}$	$-ve \quad 0 \quad +ve$

nature table shows point is a minimum ***M1A1***

[5 marks]

(d) $\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$ ***A1***

$2 \times 3e^{3x} + x \times 3^2 e^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ ***M1A1***

point is $\left(-\frac{2}{3}, -\frac{2}{3e^2}\right)$ ***A1***

x	$-\frac{2}{3}$
$\frac{d^2y}{dx^2}$	$-ve \quad 0 \quad +ve$

since the curvature does change (concave down to concave up) it is a point of inflection ***R1***

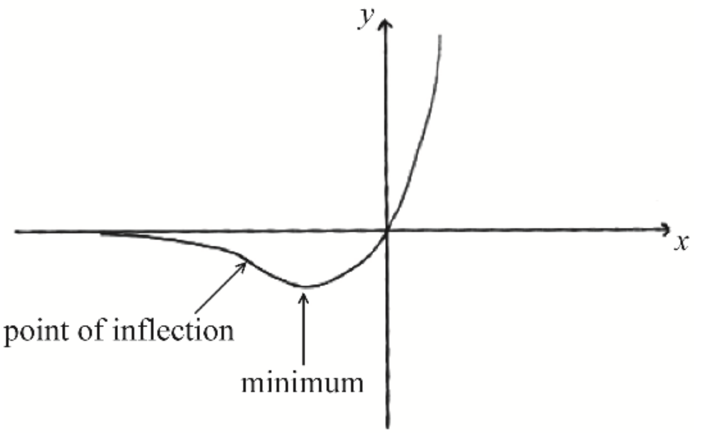
Note: Allow 3rd derivative is not zero at $-\frac{2}{3}$

[5 marks]

continued…

Question 11 continued

(e)



(general shape including asymptote and through origin) ***A1***
showing minimum and point of inflection ***A1***

Note:	Only indication of position of answers to (c) and (d) required, not coordinates.

[2 marks]

Total [21 marks]

12. (a) (i) **METHOD 1**

$\frac{v_{n+1}}{v_n} = \frac{2^{u_{n+1}}}{2^{u_n}}$ ***M1***

$= 2^{u_{n+1}-u_n} = 2^d$ ***A1***

METHOD 2

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}}$ ***M1***

$= 2^d$ ***A1***

(ii) 2^a ***A1***

Note: Accept 2^{u_1}.

(iii) **EITHER**

v_n is a GP with first term 2^a and common ratio 2^d

$v_n = 2^a(2^d)^{(n-1)}$

OR

$u_n = a + (n-1)d$ as it is an AP

THEN

$v_n = 2^{a+(n-1)d}$ ***A1***

[4 marks]

(b) (i) $S_n = \frac{2^a\left((2^d)^n - 1\right)}{2^d - 1} = \frac{2^a\left(2^{dn} - 1\right)}{2^d - 1}$ ***M1A1***

Note: Accept either expression.

(ii) for sum to infinity to exist need $-1 < 2^d < 1$ ***R1***

$\Rightarrow \log 2^d < 0 \Rightarrow d \log 2 < 0 \Rightarrow d < 0$ ***(M1)A1***

Note: Also allow graph of 2^d.

(iii) $S_\infty = \frac{2^a}{1-2^d}$ ***A1***

continued…

Question 12 continued

(iv) $\dfrac{2^a}{1-2^d}=2^{a+1}\Rightarrow\dfrac{1}{1-2^d}=2$ ***M1***

$\Rightarrow 1=2-2^{d+1}\Rightarrow 2^{d+1}=1$

$\Rightarrow d=-1$ ***A1***

[8 marks]

(c) **METHOD 1**

$w_n=pq^{n-1},\ z_n=\ln pq^{n-1}$ ***(A1)***

$z_n=\ln p+(n-1)\ln q$ ***M1A1***

$z_{n+1}-z_n=(\ln p+n\ln q)-(\ln p+(n-1)\ln q)=\ln q$

which is a constant so this is an AP

(with first term $\ln p$ and common difference $\ln q$)

$\sum_{i=1}^{n} z_i=\dfrac{n}{2}\left(2\ln p+(n-1)\ln q\right)$ ***M1***

$=n\left(\ln p+\ln q^{\left(\frac{n-1}{2}\right)}\right)=n\ln\left(pq^{\left(\frac{n-1}{2}\right)}\right)$ ***(M1)***

$=\ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***A1***

METHOD 2

$\sum_{i=1}^{n} z_i=\ln p+\ln pq+\ln pq^2+\ldots+\ln pq^{n-1}$ ***(M1)A1***

$=\ln\left(p^n q^{(1+2+3+\ldots+(n-1))}\right)$ ***(M1)A1***

$=\ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***(M1)A1***

[6 marks]

Total [18 marks]

13. (a) $\overrightarrow{OP} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$\overrightarrow{OQ} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ ***(M1)***

$\overrightarrow{PQ} = \boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k} - \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$= (1 - \lambda + \mu)\boldsymbol{i} + (-1 - \lambda - \mu)\boldsymbol{j} + (-4 - \lambda + 2\mu)\boldsymbol{k}$ ***A1***

[2 marks]

(b) **METHOD 1**

use of scalar product ***M1***

perpendicular to $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ gives

$(1 - \lambda + \mu) + (-1 - \lambda - \mu) + (-4 - \lambda + 2\mu) = 0$

$\Rightarrow -3\lambda + 2\mu = 4$ ***A1***

perpendicular to $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ gives

$(1 - \lambda + \mu) - (-1 - \lambda - \mu) + 2(-4 - \lambda + 2\mu) = 0$

$\Rightarrow -2\lambda + 6\mu = 6$ ***A1***

solving simultaneous equations gives $\lambda = -\frac{6}{7}, \ \mu = \frac{5}{7}$ ***A1A1***

METHOD 2

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***M1A1***

$\overrightarrow{PQ} = a(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$1 - \lambda + \mu = 3a$

$-1 - \lambda - \mu = -a$ ***A1***

$-4 - \lambda + 2\mu = -2a$

solving simultaneous equations gives $\lambda = -\frac{6}{7}, \ \mu = \frac{5}{7}$ ***A1A1***

[5 marks]

(c) $\overrightarrow{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ ***A1***

shortest distance $= \left|\overrightarrow{PQ}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[3 marks]

(d) **METHOD 1**

vector perpendicular to Π is given by vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$ ***(R1)***

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***(M1)A1***

so equation of Π is $3x - y - 2z + d = 0$

through $(1, 2, 3) \Rightarrow d = 5$ ***M1***

so equation is $3x - y - 2z + 5 = 0$ ***A1***

continued…

Question 13 continued

METHOD 2

from part (b) $\overrightarrow{\text{PQ}} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ is a vector perpendicular to Π ***R1A2***

so equation of Π is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + c = 0$

through $(1, 2, 3) \Rightarrow c = \frac{30}{7}$ ***M1***

so equation is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + \frac{30}{7} = 0$ $(3x - y - 2z + 5 = 0)$ ***A1***

Note: Allow other methods *ie* via vector parametric equation.

[5 marks]

(e) $\overrightarrow{\text{OT}} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \eta(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$
$T = (2 + 3\eta, 1 - \eta, -1 - 2\eta)$ lies on Π implies
$3(2 + 3\eta) - (1 - \eta) - 2(-1 - 2\eta) + 5 = 0$ ***M1***
$\Rightarrow 12 + 14\eta = 0 \Rightarrow \eta = -\frac{6}{7}$ ***A1***

Note: If no marks awarded in (d) but correct vector product calculated in (e) award ***M1A1*** in (d).

[2 marks]

(f) $\left|\overrightarrow{\text{BT}}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[2 marks]

(g) they agree ***A1***

Note: FT is inappropriate here.

$\overrightarrow{\text{BT}}$ is perpendicular to both Π and l_2
so its length is the shortest distance between Π and l_2 which is the shortest distance between l_1 and l_2 ***R1***

[2 marks]

Total [21 marks]